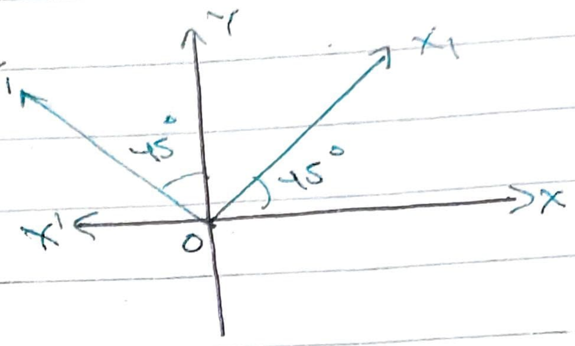
4) i) Transform the axes inclined at 45° to the original axes the equation $x^2 - y^2 = a$

Solution:



given equation is

$x^2 - y^2 = a^2$ —— (1)

Let O be the origin and OX and OY be the coordinate axes.

Suppose the axes is rotated through an angle α, the new axes becomes OX_1 and OY_1. We have

$x = x_1 \cos\alpha - y_1 \sin\alpha$

$y = x_1 \sin\alpha + y_1 \cos\alpha$.

given $\alpha = 45^\circ$.

$\therefore \; x = x_1 \cos 45^\circ - y_1 \sin 45^\circ$

$= x_1 \cdot \frac{1}{\sqrt{2}} - y_1 \cdot \frac{1}{\sqrt{2}}$.

$x = \frac{1}{\sqrt{2}}(x_1 - y_1)$

$$y = x_1 \sin\alpha + y_1 \cos\alpha$$
$$= x_1 \sin 45^\circ + y_1 \cos 45^\circ$$
$$= x_1 \cdot \frac{1}{\sqrt{2}} + y_1 \cdot \frac{1}{\sqrt{2}}$$

$$y = \frac{1}{\sqrt{2}}(x_1 + y_1)$$

putting the value of x and y in eq^n (1) we get

$$\left\{\frac{1}{\sqrt{2}}(x_1 - y_1)\right\}^2 - \left\{\frac{1}{\sqrt{2}}(x_1 + y_1)\right\}^2 = a^2$$

or, $\frac{1}{2}(x_1^2 + y_1^2 - 2x_1y_1) - \frac{1}{2}(x_1^2 + y_1^2 + 2x_1y_1) = a^2$.

or, $\frac{1}{2}(x_1^2 + y_1^2 - 2x_1y_1 - x_1^2 - y_1^2 - 2x_1y_1) = a^2$.

or, $-\frac{4x_1y_1}{2} = a^2$

or, $-2x_1y_1 = a^2$.

or, $2x_1y_1 + a^2 = 0$

Hence, Required transformed equation is
$$\boxed{2x_1y_1 + a^2 = 0}$$

ii) If the axes be turned through 45°, find the transformed form of $3x^2 + 3y^2 + 2xy = 2$.

solution.

given equation of curve is
$$3x^2 + 3y^2 + 2xy = 2 \quad \text{——— (1)}.$$

Let O be the origin and OX and OY are the coordinate axes.
Let the axes be rotated through an angle 45° the new axes becomes OX_1 and OY_1 then we have

$$x = x_1\cos\alpha - y_1\sin\alpha$$
$$= x_1\cos45° - y_1\sin45°$$
$$= \frac{x_1}{\sqrt{2}} - \frac{y_1}{\sqrt{2}}$$
$$x = \frac{1}{\sqrt{2}}(x_1 - y_1)$$

& $$y = x_1\sin\alpha + y_1\cos\alpha$$
$$= x_1\sin45° + y_1\cos45°$$
$$= \frac{1}{\sqrt{2}}(x_1 + y_1)$$

putting the value of x_1 and y_1 in eq^n (1) we get

$$3\left\{\frac{1}{\sqrt{2}}(x_1 - y_1)\right\}^2 + 3\left\{\frac{1}{\sqrt{2}}(x_1 + y_1)\right\}^2 + 2\frac{1}{\sqrt{2}}(x_1 - y_1)\frac{1}{\sqrt{2}}(x_1 + y_1) = 2$$

or, $\frac{3}{2}(x_1^2 + y_1^2 - 2x_1y_1) + \frac{3}{2}(x_1^2 + y_1^2 + 2x_1y_1) + (x_1^2 - y_1^2) = 2$

or, $\frac{3}{2}(x_1^2 + y_1^2 - 2x_1y_1 + x_1^2 + y_1^2 + 2x_1y_1) + (x_1^2 - y_1^2) = 2$.

or, $\frac{3}{2}(2x_1^2 + 2y_1^2) + (x_1^2 - y_1^2) = 2$

or, $3x_1^2 + 3y_1^2 + x_1^2 - y_1^2 = 2$

$\therefore$ $4x_1^2 + 2y_1^2 = 2$

$2x_1^2 + y_1^2 = 1$

Hence, Required transformation eq^n is

$\boxed{2x_1^2 + y_1^2 = 1}$

iii) Transform the equation $y^2 - 4x + 4y = 0$ to axes inclined at $45°$ to the original axes.

Solution:

given equation is

$y^2 - 4x + 4y = 0$ —— ①

Let O be the origin and OX and OY be coordinate axes.
suppose axes be rotated through an angle α, the new axes becomes OX_1 & OY_1
then, we have

$x = x_1 \cos\alpha - y_1 \sin\alpha$
$= x_1 \cos 45° - y_1 \sin 45°$
$= \frac{1}{\sqrt{2}}(x_1 - y_1)$

& $y = x_1 \sin\alpha + y_1 \cos\alpha$
$= \frac{1}{\sqrt{2}}(x_1 + y_1)$

putting the value of x and y in eq^n ① we get

$\left\{\frac{1}{\sqrt{2}}(x_1 + y_1)\right\}^2 - \frac{4}{\sqrt{2}}(x_1 - y_1) + \frac{4}{\sqrt{2}}(x_1 + y_1) = 0.$

or, $\frac{1}{2}(x_1^2 + y_1^2 + 2x_1y_1) + \frac{4}{\sqrt{2}}(-\cancel{x_1} + y_1 + \cancel{x_1} + y_1) = 0.$

or, $\frac{1}{2}(x_1^2 + y_1^2 + 2x_1y_1) + \frac{4}{\sqrt{2}}(2y_1) = 0.$

or, $\frac{1}{2}(x_1 + y_1)^2 + 4\sqrt{2}\, y_1 = 0.$

Hence, Required transformed eq^n is

$\boxed{(x_1 + y_1)^2 + 8\sqrt{2}\, y_1 = 0}$

iv) Transform the equation $x^2 + 2xy\tan 2\alpha - y^2 = a^2 \sec 2\alpha$ to rectangular axes inclined at an angle α to the old rectangular axes.

solution:

given equation is

$$x^2 + 2xy\tan 2\alpha - y^2 = a^2 \sec 2\alpha \quad \text{---} (1)$$

Let O be the origin and OX and OY be the coordinate axes.

suppose axes be rotated through an angle α, the new axis becomes OX_1, OY_1 then we have

$$x = x_1 \cos\alpha - y_1 \sin\alpha$$

$$y = x_1 \sin\alpha + y_1 \cos\alpha$$

putting the value of x and y in eqn (1) we get

or, $(x_1\cos\alpha - y_1\sin\alpha)^2 + 2\tan 2\alpha\,(x_1\cos\alpha - y_1\sin\alpha)(x_1\sin\alpha + y_1\cos\alpha) - (x_1\sin\alpha + y_1\cos\alpha)^2 = a^2\sec 2\alpha.$

or, $x_1^2\cos^2\alpha + y_1^2\sin^2\alpha - 2x_1y_1\sin\alpha\cdot\cos\alpha + 2\tan 2\alpha\,(x_1^2\sin\alpha\cdot\cos\alpha + x_1y_1\cos^2\alpha - x_1y_1\sin^2\alpha - y_1^2\sin\alpha\cdot\cos\alpha) - x_1^2\sin^2\alpha - y_1^2\cos^2\alpha - 2x_1y_1\sin\alpha\cdot\cos\alpha = a^2\sec 2\alpha.$

or, $x_1^2(\cos^2\alpha - \sin^2\alpha) - y_1^2(\cos^2\alpha - \sin^2\alpha) - x_1y_1\sin 2\alpha + 2\tan 2\alpha\cdot\sin\alpha\cdot\cos\alpha\cdot x_1^2 + 2x_1y_1\tan 2\alpha\cdot\cos^2\alpha - 2x_1y_1\tan 2\alpha\cdot\sin^2\alpha - 2y_1^2\tan 2\alpha\cdot\sin\alpha\cdot\cos\alpha - x_1y_1\sin 2\alpha = a^2\sec 2\alpha.$

or, $x_1^2 \cos 2\alpha - y_1^2 \cos 2\alpha - 2x_1y_1 \sin 2\alpha + \tan 2\alpha \cdot \sin 2\alpha \cdot x_1^2$
$+ 2x_1y_1 \tan 2\alpha \cdot (\cos^2\alpha - \sin^2\alpha) - y_1^2 \sin 2\alpha \cdot \tan 2\alpha$
$= a^2 \sec 2\alpha.$

or, $\cos 2\alpha (x_1^2 - y_1^2) - 2x_1y_1 \sin 2\alpha + \frac{\sin^2 2\alpha}{\cos 2\alpha} \cdot x_1^2 +$
$2x_1y_1 \cdot \tan 2\alpha \cdot \cos 2\alpha - y_1^2 \frac{\sin^2 2\alpha}{\cos 2\alpha} = a^2 \sec 2\alpha$

or, $\cos 2\alpha (x_1^2 - y_1^2) + \frac{\sin^2 2\alpha}{\cos 2\alpha}(x_1^2 - y_1^2) - 2x_1y_1 \sin 2\alpha$
$+ 2x_1y_1 \sin 2\alpha \quad = a^2 \sec 2\alpha$

or, $\cos 2\alpha (x_1^2 - y_1^2) + \frac{\sin^2 2\alpha}{\cos 2\alpha}(x_1^2 - y_1^2) = a^2 \sec 2\alpha$

or, $(x_1^2 - y_1^2)\left(\cos 2\alpha + \frac{\sin^2 2\alpha}{\cos 2\alpha}\right) = a^2 \sec 2\alpha$

or, $(x_1^2 - y_1^2)\left(\frac{\cos^2 2\alpha + \sin^2 2\alpha}{\cos 2\alpha}\right) = a^2 \sec 2\alpha.$

or, $(x_1^2 - y_1^2) \times \frac{1}{\cos 2\alpha} = a^2 \sec 2\alpha$

or, $x_1^2 - y_1^2 = a^2 \sec 2\alpha \cdot \cos 2\alpha$

$\therefore \quad x_1^2 - y_1^2 = a^2$

Hence, Required transformed eq^n is

$$\boxed{x_1^2 - y_1^2 = a^2}$$

v) Transform to axes inclined at 30° to the original axes the equation
$x^2 + 2\sqrt{3}\,xy - y^2 = 2a^2$

Solution:

given equation is
$$x^2 + 2\sqrt{3}\,xy - y^2 = 2a^2 \quad \text{---------} \quad (1)$$

Let O be the origin and OX & OY be the coordinate axes.
suppose axes be rotated through an angle 30° the new axes becomes OX_1 & OY_1 then we have

$$x = x_1\cos\alpha - y_1\sin\alpha$$
$$= x_1\cos 30^\circ - y_1\sin 30^\circ$$
$$= x_1\frac{\sqrt{3}}{2} - y_1\frac{1}{2}$$

$$x = \frac{1}{2}(\sqrt{3}x_1 - y_1)$$

&

$$y = x_1\sin\alpha + y_1\cos\alpha$$
$$= x_1\sin 30^\circ + y_1\cos 30^\circ$$
$$= x_1\frac{1}{2} + y_1\frac{\sqrt{3}}{2}.$$

$$y_1 = \frac{1}{2}(x_1 + \sqrt{3}y_1)$$

putting the value of x and y in eq^n (1) we get

$$\left\{\frac{1}{2}(\sqrt{3}x_1 - y_1)\right\}^2 + 2\sqrt{3}\frac{1}{4}(\sqrt{3}x_1 - y_1)(x_1 + \sqrt{3}y_1)$$
$$- \left\{\frac{1}{2}(x_1 + \sqrt{3}y_1)\right\}^2 = 2a^2$$

or, $\frac{1}{4}(3x_1^2 + y_1^2 - 2\sqrt{3}x_1y_1) + \frac{\sqrt{3}}{2}(\sqrt{3}x_1^2 + 3x_1y_1 - x_1y_1 - \sqrt{3}y_1^2)$

$- \frac{1}{4}(x_1^2 + 3y_1^2 + 2\sqrt{3}x_1y_1) = 2a^2$

or, $\frac{1}{4}(3x_1^2 + y_1^2 - 2\sqrt{3}x_1y_1 - x_1^2 - 3y_1^2 - 2\sqrt{3}x_1y_1)$

$+ \frac{\sqrt{3}}{2}\{\sqrt{3}(x_1^2 - y_1^2) + 2x_1y_1\} = 2a^2$

or, $\frac{1}{4}(2x_1^2 - 2y_1^2 - 4\sqrt{3}x_1y_1) + \frac{3}{2}(x_1^2 - y_1^2) + \sqrt{3}x_1y_1 = 2a^2$

or, $\frac{1}{2}(x_1^2 - y_1^2 - 2\sqrt{3}x_1y_1) + \frac{3}{2}(x_1^2 - y_1^2) + \sqrt{3}x_1y_1 = 2a^2$

or, $\frac{1}{2}(x_1^2 - y_1^2 - 2\sqrt{3}x_1y_1 + 3x_1^2 - 3y_1^2 + 2\sqrt{3}x_1y_1) = 2a^2$

or, $\frac{1}{2}(4x_1^2 - 4y_1^2) = 2a^2$

or, $2(x_1^2 - y_1^2) = 2a^2$

or, $x_1^2 - y_1^2 = a^2$

Hence, Required transformed eqn is

$\boxed{x_1^2 - y_1^2 = a^2}$

vi) Find the changed form of the equation

$x^2 + y^2 + 2(x - y) + 1 = 0$

when the axes (rectangular) are turned through $45°$ the origin remaining fixed.

Solution:

given equation is

$x^2+y^2+2(x-y)+1=0$ —— ①

Let O be the origin and OX & OY be the coordinate axes
suppose the axes be rotated through an angle 45°, the new axes becomes OX_1 & OY_1
then we have

$x = x_1\cos\alpha - y_1\sin\alpha$
$= x_1\cos 45^\circ - y_1\sin 45^\circ$
$= \frac{1}{\sqrt{2}}(x_1-y_1)$

$y = x_1 \sin\alpha + y_1\cos\alpha$
$= \frac{1}{\sqrt{2}}(x_1+y_1)$

Putting the value of x and y in eq^n ①.

$\left\{\frac{1}{\sqrt{2}}(x_1-y_1)\right\}^2 + \left\{\frac{1}{\sqrt{2}}(x_1+y_1)\right\}^2 + \frac{2}{\sqrt{2}}(x_1-y_1-x_1-y_1)+1=0$

or, $\frac{1}{2}(x_1-y_1)^2 + \frac{1}{2}(x_1+y_1)^2 + \sqrt{2}(-2y_1)+1=0$

or, $\frac{1}{2}(x_1^2+y_1^2-2x_1y_1+x_1^2+y_1^2+2x_1y_1) - 2\sqrt{2}y_1+1=0$

or, $x_1^2+y_1^2-2\sqrt{2}y_1+1=0$

Hence, Required transformed eq^n is

$$\boxed{x_1^2+y_1^2-2\sqrt{2}y_1+1=0.}$$